THE THIRTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth summer meeting and the fifth colloquium of the Society were held at Yale University during the week September 3–8, 1906. The following forty-six members were in attendance:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Dr. J. E. Clarke, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Mr. H. N. Davis, Professor L. P. Eisenhart, Professor W. B. Fite, Professor A. S. Gale, Professor C. N. Haskins, Dr. L. I. Hewes, Professor Edward Kasner, Professor O. D. Kellogg, Dr. W. R. Longley, Mr. E. B. Lytle, Professor T. E. McKinney, Professor James McMahon, Professor H. P. Manning, Professor Max Mason, Mr. E. A. Miller, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. R. G. D. Richardson, Miss S. F. Richardson, Professor H. L. Rietz, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Dr. C. H. Sisam, Professor C. S. Slichter, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor W. E. Story, Professor E. B. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. Young.

The colloquium, of which a separate report appears in the BULLETIN, opened on Wednesday morning. The summer meeting proper extended through two sessions on Monday and a morning session on Tuesday. On Tuesday afternoon the visitors were conducted through the grounds and buildings of the University. The generous hospitality extended to the Society throughout the meeting by the University and its officers will long be remembered with grateful appreciation.

At the opening session on Monday morning Professor E. H. Moore presided. President W. F. Osgood occupied the chair during the remaining sessions. The Council announced the election of the following persons to membership in the Society: Professor William Beebe, Yale University; Mr. J. B. Clarke, Polytechnic High School, San Francisco; Dr. E. C. Colpitts,

Cornell University; Brother Constantius, Christian Brothers College, St. Louis; Professor G. W. Droke, University of Arkansas; Mr. R. M. Ginnings, State Normal School, Kirksville, Mo.. Professor Harriet E. Glazier, Western College for Women; Professor C. O. Gunther, Stevens Institute of Technology; Mr. W. A. Hurwitz, University of Missouri; Dr. G. O. James, Washington University; Mr. B. F. Johnson, State Normal School, Cape Girardeau, Mo.; Mr. E. B. Morrow, Princeton University; Mr. G. B. Obear, Brown University; Dr. F. M. Pedersen, College of the City of New York; Professor G. A. Rose, Hardin College; Mr. R. L. Short, Chicago, Ill.; Miss Betty Trier, Mount Holyoke College; President J. W. Withers, Teachers College, St. Louis. Thirteen applications for membership in the Society were received.

A committee consisting of Professors Bôcher, E. B. Van Vleck, and Townsend was appointed by the President to prepare and report to the Council at the October meeting a list of nominations for officers and other members of the Council to be elected at the annual meeting in December. Steps were also taken toward amending the Constitution to include the Editorial Committee of the *Transactions* in the membership of the Council.

The following papers were read at the summer meeting:

(1) Mr. A. R. SCHWEITZER: "Systems of axioms for projective geometry."

(2) Mr. A. R. SCHWEITZER: "Concerning abstract geometric relations" (preliminary report).

(3) Professor O. D. KELLOGG: "The behavior on the boundary of harmonic functions of a region."

(4) Mr. F. R. SHARPE: "The motion of a viscous gas."

(5) Professor R. D. CARMICHAEL: "Multiply perfect numbers of three different primes."

(6) Professor LUDWIG STICKELBERGER: "Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören."

(7) Professor W. B. FITE: "Irreducible linear homogeneous groups whose orders are powers of a prime."

(8) Dr. ARTHUR RANUM: "The group of classes of congruent matrices and its application to the group of isomorphisms of any abelian group."

(9) Dr. R. G. D. RICHARDSON: "On the reduction of multiple integrals (second paper)."

(10) Mr. G. D. BIRKHOFF: "On a certain class of sets of normed orthogonal functions."

(11) Dr. W. B. CARVER: "Associated configurations of the Cayley-Veronese class."

(12) Professor L. E. DICKSON: "On commutative linear algebras in which division is always uniquely possible."

(13) Professor L. E. DICKSON: "Uniform definitions of the abstract forms of the various known systems of linear groups."

(14) Professor L. E. DICKSON: "Criteria for the irreducibility of functions in a finite field."

(15) Professor L. E. DICKSON: "On the theory of equations in a modular field."

(16) Professor JAMES MCMAHON: "The differential geometry of the general vector field" (preliminary report).

(17) Dr. W. A. MANNING: "A note on transitive groups."

(18) Dr. C. H. SISAM: "On systems of conics lying on surfaces of the third, fourth, and fifth orders."

(19) Professor VIRGIL SNYDER: "Plane quintic curves which possess a group of linear transformations."

(20) Professor MAX MASON: "The expansion of an arbitrary function in terms of normal functions."

(21) Professor MAX MASON: "The boundary value problems of differential equations of hyperbolic type."

(22) Professor EDWARD KASNER: "An inverse problem of dynamics."

(23) Professor EDWARD KASNER: "The geometry of dynamical trajectories."

(24) Professor J. W. YOUNG: "General theory of approximation by functions with a given number of parameters."

(25) Professor J. I. HUTCHINSON: "On loci the coordinates of whose points are abelian functions of three parameters."

(26) Professor L. P. EISENHART: "Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of another."

(27) Dr. H. B. LEONARD: "On the factoring of composite hypercomplex number systems."

(28) Professor FRANK MORLEY: "Reflexive geometry."

(29) Professor G. A. MILLER: Generalization of the groups of genus zero."

(30) Professor E. B. WILSON: "On divergence and curl."

(31) Professor E. B. WILSON: "Oblique reflections and unimodular strains."

(32) Professor E. B. WILSON: Double products and strains in n dimensions."

(33) Professor F. R. MOULTON: "A class of three dimensional periodic orbits in the problem of three bodies, with applications to the lunar theory."

(34) Professor OSKAR BOLZA: "Weierstrass's theorem and Kneser's theorem on transversals for the most general case of an extremum of a simple definite integral."

Professor Stickelberger's paper was communicated to the Society through Professor E. H. Moore. In the absence of the authors, Mr. Sharpe's paper was presented by Professor McMahon, Dr. Ranum's by Professor Moore, Dr. Carver's by Professor Snyder, Professor Hutchinson's by Professor Fite, and the papers of Professor Carmichael, Professor Stickelberger, Professor Dickson, Dr. Manning, Dr. Leonard, Professor Morley, Professor Miller, Professor Wilson, Professor Moulton and Professor Bolza were presented by title.

The third and fourth papers of Professor Dickson and the paper of Dr. Manning appeared in full in the October BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer constructed systems of axioms for projective geometry of n dimensions ($n = 1, 2, 3, \cdots$), using two undefined symbols, viz., the element point and relation K_n between two ordered $(n+2)$-ads of points. Under the respective systems the relation K_n is symmetric and transitive, *i. e.*, $(A_1A_2\cdots A_{n+2})K_n(B_1B_2\cdots B_{n+2})$ implies $(B_1B_2\cdots B_{n+2})K_n(A_1A_2\cdots A_{n+2})$ and the statements $(A_1A_2\cdots A_{n+2})K_n(B_1B_2\cdots B_{n+2})$, $(B_1B_2\cdots B_{n+2})K_n(C_1C_2\cdots C_{n+2})$ imply $(A_1A_2\cdots A_{n+2})K_n(C_1C_2\cdots C_{n+2})$. The existence of an ordered $(n+2)$-ad implies no geometric axiom except the existence of points. Systems III, IV, $\cdots$ are sufficient for 3, 4, $\cdots$ dimensional projective geometry respectively. Systems, I, II, $\cdots$ are readily extended to higher dimensions. In each of the preceding systems, collinearity, separation, etc., are defined in terms of the corresponding relation K_n.

The above sets of axioms may be considered the projective analogues of the author's systems of descriptive axioms* for

*Cf. abstract No. 6, BULLETIN, June, 1906, p. 438; for n-ad read $(n+1)$-ad. It should also there be stated that under the axioms the K_n is symmetric and transitive in the sense indicated above.

n dimensions ($n = 1, 2, 3, \cdots$) involving the two undefined elements point and the symmetric, transitive relation K_n between two ordered $(n + 1)$-ads of points. A result of both investigations is that it is possible to construct axioms for n dimensions, ($n = 1, 2, 3, \cdots$), which may be either descriptive or projective by means of two undefined symbols point and a transitive, symmetric relation; and in n-dimensional space this relation may be either linear, or planar, $\cdots$, or n-dimensional.

2. In his second paper Mr. Schweitzer investigates the kinds of undefined relations between points which may be used in the construction of geometric axioms, for example: 1) $\alpha R_n(\beta_1\beta_2\cdots\beta_n)$, ($n = 1, 2, 3, \cdots$), where the n-ad is ordered; 2) $[\alpha_1\cdots\alpha_n]B_n[\beta_1\beta_2]$, ($n = 1, 2, 3, \cdots$), where the dyad and n-ad are not ordered; 3) $(\alpha_1\alpha_2)S_{n+2}(\beta_1\beta_2\cdots\beta_{n+1})(n=1, 2, 3, \cdots)$, where the dyad and $(n + 1)$-ad are both ordered. In particular, the construction of descriptive axioms in terms of the relation R_n under 1) leads to essential simplifications of a treatment of geometric order given by Vahlen in his "Abstrakte Geometrie." This is shown in a note which will appear in the BULLETIN.

3. The object of Professor Kellogg's paper is the study of the behavior on the boundary of the derivatives of harmonic functions of a region, and allied questions. Some preliminary results for the circle are here given. The formulas of Hilbert,

$$f(s) = \int_0^1 g(t)\cot\pi(s-t)dt + \int_0^1 f(t)dt,$$

$$g(s) = -\int_0^1 f(t)\cot\pi(s-t)dt + \int_0^1 g(t)dt,$$

giving the boundary values of a harmonic function of the unit circle in terms of those of the conjugate function, still hold if we only postulate piecewise regularity of $f(s)$ and $g(s)$. By regularity we mean that a condition holds of the form $|f(s+h) - f(s)| < Ch^\alpha$, where C and α are positive constants, C as large and α as small as we please. A study of these formulas leads to the results: It is not sufficient for the continuity of the boundary values of a harmonic function that the boundary values of the conjugate function be continuous. It is sufficient that they be regular. In order that any first derivative of a

harmonic function be continuous up to and upon the boundary, it is not sufficient that the tangential derivatives on the boundary be continuous. It is sufficient that they be regular. (This last result is proved for the circle in a slightly more general form by Dini, *Acta Mathematica*, volume 25, page 221 and following.) The necessary and sufficient conditions can be brought even nearer together, but it seems scarcely worth while. The extension of these results to more general regions will be the subject of a later report.

Another remark concerns the distinction between the conformality of the mapping of a function of a complex variable and the finiteness and non-vanishing of the derivative. That these are not the same is at once seen by an examination of the mapping given by $\zeta = z \log z$ in the neighborhood of the origin.

The paper will be offered to the *Transactions* for publication.

4. In discussing the motion of a gas it is usually assumed that every element neither gains nor loses heat, and consequently the pressure varies as some power of the density so long as we consider elements on the same stream line. In Mr. Sharpe's paper the adiabatic assumption is replaced by a new equation obtained by considering the total flow of energy, molar and molecular, which includes the effects of viscosity and conduction of heat. It is shown that this equation agrees with a similar one deduced from the kinetic theory of gases, and it is applied to the following problems: 1) heated air rising up a chimney, 2) gas flowing between two smooth parallel planes, 3) gas flowing between two rough parallel planes, 4) gas surrounding a rotating circular cylinder and inside a fixed concentric cylinder.

5. In this paper Professor Carmichael completes the investigation of the existence of multiply perfect numbers of (only) three different prime factors, showing that the only such numbers are $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$, each of multiplicity 3. In a footnote he also points out that there exists no multiply perfect number which is the power of a prime, and that all perfect numbers of (only) two different primes are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. The paper will be published in the *Annals of Mathematics*.

6. Professor Stickelberger gives a very simple proof of a lemma forming the basis of the article by Professor Loewy,

"Über die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören" (*Transactions*, volume 6 (1905), pages 504–533). Moreover, there results an extension of Loewy's theorem. The paper will appear in the *Transactions*.

7. Certain irreducible linear homogeneous groups of different degrees are simply isomorphic; whereas, on the other hand, all the irreducible groups with which some groups are simply isomorphic are of the same degree. The question thus suggested as to the connection between the degree of an irreducible group and its abstract group properties forms the subject of Professor Fite's paper. The discussion is limited to groups whose orders are powers of a prime.

8. With respect to the n^2 moduli $p^{\alpha_{ij}}(i,j=1,\cdots n;\ p=\text{a prime})$ two matrices (l_{ij}) and (m_{ij}) are said to be congruent, if $l_{ij} \equiv m_{ij}$, mod $p^{\alpha_{ij}}$. Dr. Ranum proves that the composition of classes of congruent matrices is possible if, and only if, the moduli satisfy the condition $a_{ij} + a_{jk} \geqq a_{ik}\,(i, j, k = 1, \cdots, n)$ and the elements l_{ij} are divisible by $p^{\alpha_{ij}}$, where α_{ij} is the greatest of the $2n + 1$ integers 0, $a_{ik} - a_{jk}$, $a_{kj} - a_{ki}\,(k = 1, \cdots, n)$. If these conditions are satisfied, the totality of classes of matrices whose determinants are prime to p form a group, of which all other groups satisfying the same condition are subgroups. The order, invariant matrices, and composition series of this group, and its largest invariant subgroup of order a power of p, are found. The application of a special group of this kind to the group of isomorphisms of any abelian group is an amplification of a paper read before the San Francisco Section at its February meeting, entitled "A new kind of congruence group, etc." (abstract in BULLETIN, May, 1906, page 375).

9. In a former paper (April, 1906), Dr. Richardson showed that in all cases the existence of the double integral and the iterated integral is a sufficient condition for their equality. The present paper discusses the corresponding problem which arises when either the double integral or the iterated integral exists. In the *Journal de Mathématiques*, series 5, volume 5 (1899), de la Vallée-Poussin derived the relation

$$1)\quad \bar{\int_T} f dT \geqq \underline{\int_y} dy \bar{\int_z} m f dx \geqq \underline{\int_y} dy \underline{\int_x} m f dx \geqq \underline{\int_T} f dT \qquad (f \geqq 0),$$

where T is a portion of the plane bounded by a regular curve and the minimum function mf is defined at each point as the minimum of f in an infinitesimal area about that point. The introduction of the minimum function is shown to be unnecessary and relation 1) is true if f is substituted for mf. If the double integral exists, this becomes

$$\int_T f dT = \underline{\int}_y dy \bar{\int}_x f dx = \underline{\int}_y dy \underline{\int}_x f dx.$$

Other relations are derived under the hypothesis that the iterated integral alone exists. For the generalized integral introduced by Professor Pierpont, the theorems are valid; and if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m+n$, m, and n dimensions respectively, the discussion applies. Finally, the theorems are extended to the case where the field of integration is unlimited.

10. Mr. Birkhoff's paper deals with certain infinite sets of normed orthogonal functions, $\phi_1(x)$, $\phi_2(x)$, $\cdots$, *i. e.*, sets of functions ϕ such that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \begin{matrix} 0, & (i \neq j) \\ 1, & (i = j) \end{matrix};$$

such a set is defined as the set of normed solutions of an equation

$$\phi(x) = \lambda \int_0^1 k(x, t)\phi(t)dt,$$

where λ is a parameter and where 1) $k(x, t)$ is defined for $0 \leqq x, t \leqq 1$ and is real, continuous, and symmetric in x and t; 2) $k(x, t)$ has continuous partial derivatives in x, t of first and second orders, except for $x = t$, where there is the discontinuity of a finite jump, at least in the first partial derivatives.

The principal theorems deal with the possibility of representing a prescribed function $f(x)$ on the interval (0, 1) as a series in the functions ϕ. The recent results of Fredholm, Hilbert and Schmidt in this direction are fundamental for the standpoint and theorems of the paper.

11. Beginning with the idea of $n+2$ points in S_n associated with $n+2$ S_{n-1}'s, one may define a $C^{n+1}_{n+2,r}$ associated with a

given $C^n_{n+2,r}$ (the notation being that used by Dr. Carver in an earlier paper, "On the Cayley-Veronese class of configurations," *Transactions*, October, 1905). A method is given for constructing the polar point of a $C^{n-1}_{n+1,r}$ with respect to a $C^n_{n+1,r}$; and also for constructing a polar S_{k-1} of a $C^{n-k}_{n+1,r}$ with respect to a $C^n_{n+1,r}$.

12. In his first paper, Professor Dickson considers commutative linear algebras in $2n$ units, with coördinates in a general field F, such that n of the units define a subalgebra forming a field $F(J)$. The elements of the algebra may be exhibited in the form $A + BI$, although the algebra is not binary. Algebras VIII and X, on four and six units, respectively, given in the July number of the *Transactions*, may now be given a more luminous form (valid for n arbitrary), in which the product of any two elements is

$$(A + BI)(X + YI) = AX + B'Y'J + (AY + BX)I,$$

where $B' = B(J')$, J' being a second root of the normal equation satisfied by J. If the constant term of the latter equation is a not-square in F, division is uniquely possible in the algebra.

By an investigation of the general commutative linear algebra in $2n$ units in which division is unique, we are led to algebras with the law of multiplication $(A + BI)(X + YI) = R + SI$, where

$$R = AX + L(BY - B'Y') + JB'Y',$$

$$S = AY + BX + U(BY - B'Y').$$

For $n = 2$ and $J^2 - C_1J + C_2 = 0$ the conditions for unique division are

$$L = J^2U'^2/4C_2 - \tfrac{1}{4}U^2, \quad C_2 = \text{not-square in } F,$$

$$(J'U^2 + JU'^2 - 4C_2)^2 - 4C_2U^2U'^2 = \text{square in } F.$$

Aside from the case $U = 0$ (giving the first algebra above), these algebras are equivalent, under linear transformation, in sets of four, while the identity is the only transformation into itself of such an algebra. The article will appear in the *Transactions*.

13. In his second paper, Professor Dickson defines abstractly various systems of linear groups with coëfficients in an arbitrary field F, viz., the general linear group, the abelian and hyperabelian groups, and the systems of linear groups defined by a quadratic or a hermitian form. The simplicity of the sets of generational relations is due to the choice of the generators (all of period p when F has modulus p; all non-periodic when F is non-modular). The great majority of the relations express the commutators of each pair of generators. The defining relations have intimate, but not entirely obvious, connections with Lie's commutator relations for the corresponding continuous groups. Two papers incorporating the results for the general linear and the abelian groups have been offered for publication in the *Quarterly Journal of Mathematics*.

16. A systematic study of the geometry of the general vector field without immediate reference to physical applications would be of considerable mathematical interest, and, if carried far enough, must ultimately throw new light on various physical problems. In Professor McMahon's preliminary paper, the vector lines of the field are viewed as a family of twisted curves, which may or may not have a family of orthogonal surfaces. The jacobian determinant of u, v, w, the three rectangular components of the field at (x, y, z), is studied as a fundamental invariant. The linear transformation of any vector (l, m, n), by using a matrix composed of the elements of the jacobian of (u, v, w), produces another vector that bears important relations to the field. The conjugate and reciprocal matrices produce other transformed vectors whose invariant relations are studied, and interpreted geometrically. The differential equations of various other invariant lines are derived, such as the locus of inflexions, lines along which the field vector preserves its direction or magnitude, points of maximum curvature, direction of greatest and least derivative of vector or of tensor. The properties of the auxiliary quadrics and of the general vector tube are considered.

18. Dr. Sisam discusses in this paper the properties of algebraic surfaces generated by systems of conics. A number of general properties of such surfaces are determined and a classification is made of the surfaces so generated which are of order not higher than five.

19. The purpose of Professor Snyder's paper was to find those quintic curves which are left invariant by linear transformations. The only collineations that can transform a plane quintic into itself must all have the same invariant triangle. Apart from a few obvious types and admitting one cyclic collineation, thirteen forms were found. In particular the triangular-symmetric curve has a group of order 150. The largest number of harmonic homologies in any group is 15; in two different types 5 can exist, and one has 3. The remaining types have not more than one.

20. Professor Mason gave a proof for the expansion of an arbitrary function which has a sectionally continuous first derivative in terms of normal functions of the differential equation

$$\frac{d^2y}{dx^2} + \lambda A(x)y = 0.$$

Application was made of the fact that the normal functions are solutions of certain minimum problems.

21. In Professor Mason's second paper a boundary value problem for the differential equation

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu + f$$

was treated which includes several of the boundary problems previously considered. The resolution of the problem was made to depend on an indefinite integral equation, the solution being obtained by successive substitutions.

22. The determination of the trajectories of a given positional force requires the integration of the equations of motion. The inverse problem which forms the subject of Professor Kasner's paper consists in determining the field of force when the trajectories are given. Analytically the problem is solved by a quadrature. The object of the paper is to obtain a direct geometric construction of the field. This is attained by employing the auxiliary system of curves, termed velocity curves, which the author has studied in a previous paper. This related system is decomposable in a simple way into ∞^1 subsystems, from any

one of which the field can be constructed. The problem is solved for both plane and space fields.

23. In previous papers Professor Kasner has studied the motion of a particle acted upon by an arbitrary (positional) field of force in the plane. A complete set of geometric properties of the trajectories is presented in the *Transactions* for July, 1906. The present paper discusses the general three-dimensional problem. The results are not only more complicated, as is to be expected, but also quite different in character. The osculating spheres of the twisted curve take the place of the osculating parabolas employed in the plane theory to interpret differential elements of the third order. It is shown that the osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line; and that, as the initial direction varies, the straight line generates a congruence consisting of the secants of a twisted cubic. Converse questions are considered and ultimately a characteristic set of properties is obtained.

24. Professor Young's paper considers a general problem of approximation, which he states as follows: Given a class $\mathfrak{C}_n$ of functions $f = f(x) = f(x; a_0, a_1, \cdots, a_n)$ of the real variable x and a functional operation U, such that every $U(f)$ is a fully defined function of x at every point of a finite interval (a, b) as soon as the parameters a_i are given; to determine these a_i so that the maximum of $|U(f)|$, as x varies over (a, b), shall be as small as possible. The existence of a solution, which is called a "function of approximation in $\mathfrak{C}_n$ with reference to U," is proved under very general conditions on $U(f)$. The author then assigns to the functions f a more definite form; *i. e.*, he considers a class $\mathfrak{S}_n$ of functions

$$S_n = a_0 s_0 + a_1 s_1 + \cdots + a_n s_n,$$

where the s_i are given functions of x, and seeks to determine the a_i so that the minimum of $|V(S_n) - \phi|$, as x varies over (a, b), shall be as small as possible. Here V is a distributive functional operation and ϕ a given function of x. Under certain general conditions on V, s_i, and ϕ the problem is shown to have a unique solution, and certain necessary and sufficient conditions are derived that a function in $\mathfrak{S}_n$ shall be a "function of approximation in $\mathfrak{S}_n$ with reference to ϕ and V." The

theorems thus proved form a two-fold generalization of the theorems of Tchebychev considered by the author in a paper presented to the Society at its last April meeting, and show that the properties expressed in the older theorems are characteristic of a much larger class of problems than those to which they originally applied.

25. The paper by Professor Hutchinson considers the loci in space of four dimensions determined by taking the five homogeneous coördinates of a point proportional to linearly independent theta functions of three variables of the same order and characteristic, and all odd, or all even. It is found that only a finite number of such loci exist having a given degree. By equating to zero a linear combination of the given theta functions, algebraic surfaces in space of three dimensions are determined. Various results are deduced by means of the properties of the theta functions.

26. Professor Eisenhart considers pairs of surfaces applicable to one another with the asymptotic lines of one surface corresponding to a conjugate system on the other. He finds that each of the two surfaces is an associate of a spherical surface—a surface whose gaussian curvature is positive and constant—and the two spherical surfaces are the Hazzidakis transforms of one another. Moreover, every surface associate to a spherical surface is applicable to a surface associate to the Hazzidakis transform of the spherical surface, in such a way that the asymptotic lines of the one correspond to a conjugate system of the other. To asymptotic lines on the associate surface corresponds a conjugate system with equal tangential invariants on the given spherical surface, and on its Hazzidakis transform a conjugate system with equal point invariants. When a conjugate system upon a spherical surface has equal point and tangential invariants, two pairs of applicable surfaces of the kind sought can be found by quadratures; a particular study is made of this case. When a surface is referred to its asymptotic lines, the equations of condition to be satisfied in order that the surface admit an applicable surface upon which the parametric lines form a conjugate system are of such a form that it is seen that if there are more than two such applicable surfaces there are an infinity. In this case the fundamental coefficients of the applicable surfaces can be found directly and the infinity of associate spherical surfaces are given by quadratures.

27. In the *Mathematische Annalen* (volume 39 (1891), pages 324–326) Scheffers calls the algebra $EF \equiv \epsilon_{ij} = e_i f_j = f_j e_i$ $(i = 1, \cdots, n\,;\ j = 1, \cdots, r)$ the product (compound) of the algebras $E \equiv e_1 \cdots e_n$, $F \equiv f_1 \cdots f_r$. He suggests inversely the desirability of theorems concerning the roots of the characteristic equation of the composite algebra, of a criterion for determining whether a given algebra is composite, and also of theorems concerning the division of zero in the composite algebra.

These problems are studied in a thesis prepared by Mr. Leonard in candidacy for the degree of doctor of philosophy from the University of Colorado. In § 1 it is shown that if G_E and G_F are the groups of the algebras E and F, then the transverse of the compound of these groups is a subgroup of the group G_{EF} of the compound algebra EF. In § 2 the first of the above problems is treated and the following theorem proved: If $\mu_1, \cdots, \mu_n$ are the roots of the characteristic equation of the number $X = \sum_i x_i e_i$ of the algebra E and if $\nu_1, \cdots, \nu_r$ are the roots of the characteristic equation of the number $\overline{X} = \sum_j \overline{x}_j f_j$ of the algebra F, then the roots of the corresponding number of the compound algebra are $\mu_i \nu_j (i = 1, \cdots, n\,;\ j = 1, \cdots, r)$.

The second of the above problems is treated from two distinct points of view. In § 3 the characteristic equation of the composite algebra EF being given, the characteristic equations of the factor algebras are sought. In § 4 the multiplication table of the composite algebra being given, the multiplication tables of the factor algebras are determined by means of Peirce's matrix representation of linear associative algebras. For example, by both methods the algebra

	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_1	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_2	ϵ_2	0	ϵ_4	0
ϵ_3	ϵ_3	ϵ_4	$-\epsilon_1$	$-\epsilon_2$
ϵ_4	ϵ_4	0	$-\epsilon_2$	0

is factored into the algebras

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

	f_1	f_2
f_1	f_1	f_2
f_2	f_2	$-f_1$

Thus one comes to the interesting classification of algebras into prime and composite, exactly like ordinary integers, and this leads to a theory of linear associative algebras analogous to the theory of integers.

If one of the factors is a divisor of zero, it is shown in § 5 that the composite number is a divisor of zero, and conversely.

28. Professor Morley's paper on reflexive geometry is in continuation of two which appeared in the *Transactions*, volumes 1 and 4. Given n lines of a plane, a certain curve C^{n-1} is thereby given, and the discussion of this curve in relation to the lines is the main theme. The curve C^2 is a circle, the curve C^3 a cardioid. The C^{m-1} of any m of the n lines is an osculant of the C^{n-1} of all the lines. If we build up all Jonquières curves of order $n-2$, with multiple point of order $n-3$ at a point x, and passing through the reflexions of x in the n lines, then such a curve has (projectively speaking) coincident points at infinity when x is on C^{n-1}, and it has equiangular asymptotes when x is on the centric circle. Incidentally it appears that n lines belong to and define a set of $2n-3$ lines. The case when for n odd the n lines form such a set is exceptional.

29. The object of Professor Miller's paper is to study all the groups which result if the defining equations of the groups of genus zero are generalized along certain lines. It is a continuation of the paper in volume 9 of the *Archiv der Mathematik und Physik* under the heading "The groups generated by two operators which have a common square." Some of the results are as follows: There are exactly four groups whose two generators s_1, s_2 satisfy the relations $s_1^3 = s_2^3$, $(s_1 s_2)^2 = 1$. They are the tetrahedral group, the direct product of this group and the cyclic group of orders 4 and 2, and a certain group of order 96. There are also just four groups whose two generators satisfy the conditions $s_1^2 = s_2^3$, $(s_1 s_2)^3 = 1$, while there is an infinite system of groups which satisfy the conditions $s_1^3 = s_2^3$, $(s_1 s_2^{-1})^2 = 1$. The similar generalizations of the defining equations of the octahedral and the icosahedral groups are considered and all the possible groups are completely determined. The paper has been offered to the *Transactions* for publication.

30. In the paper on divergence and curl Professor Wilson starts with the integral definitions of these differentiators. This

leads to a closer contact with physics and hydrodynamics, and brings out almost intuitively the theorems of Gauss and Stokes. The introduction of the linear vector function $\nabla\mathbf{V}$ (where $\mathbf{V}$ is a vector function of position in space) makes it possible to obtain div $\mathbf{V}$ and curl $\mathbf{V}$ as $\nabla \cdot V$ and $\nabla \times V$ respectively without the usual recourse to integral calculations relative to an infinitesimal cube or rectangle. The paper appears in the *American Journal of Science*, 1906.

31. After a short introduction giving the history of the efforts to study various subgroups of the projective group by means of involutory transformations, Professor Wilson proceeds to a systematic study of involutory projective transformations and to their special cases in various subgroups. An algebraic interest is added to the discussion by pointing out the fact that these transformations may be regarded as square roots of the idemfactor. In the main body of the paper the discussion of the composition of oblique reflection into strains and the resolution of unimodular strains into reflections is treated in detail. The analytic method employed is Gibbs's vector analysis—in particular his theory of dyadics. Among other things it is proved that: Any root of the minor idemfactor may be resolved into the product of two square roots in ∞^1 ways. The necessary and sufficient condition that a strain be resolved into two reflections is that the scalar cubic corresponding to the identical equation in the matrix representing the strain shall be a reciprocal equation. Any unimodular strain may be resolved into three reflections and the line of the first reflection is arbitrary except possibly for at most three directions.

32. In the paper on double products Professor Wilson takes up somewhat at length the generalizations of some of the questions treated in chapters V and VI of his edition of Gibbs's Vector Analysis. The present paper is perhaps the first published account of the late Professor Gibbs's ideas on multiple algebra, except for the general papers "On multiple algebra" by J. Willard Gibbs, *Proceedings of the American Association for the Advancement of Science*, volume 35 (1886), and "On products in additive fields" by the present author, Verhandlungen des III internationalen Mathematiker-Kongresses in Heidelberg, 1904. The paper will be offered for publication in the *Transactions*.

33. The paper of Professor Moulton is devoted to the consideration of a class of periodic orbits in which the mean motion of the line of nodes of the orbit of one the bodies, referred to the mean plane of the motion of the other two, is arbitrary except for the condition that it shall be commensurable with the synodic mean motion of the three bodies. The discussion involves a new treatment of Hill's linear differential equation with periodic coefficients.

As applied to the lunar theory, the paper shows how to construct expressions for the coordinates of a body having the same synodical period and mean rate of revolution of the line of nodes as observations show the moon has. These are the terms which are said to depend upon the mean motions of the sun and moon, the parallax of the moon, and the latitude of the moon.

34. Professor Bolza's paper gives an extension of Weierstrass's theorem on the expression of the total variation by means of the E-function and of Kneser's theorem on transversals to the so-called most general case of an extremum of a simple definite integral, in which it is required to minimize an integral of the form

$$I = \int_{x_0}^{x_1} f(x, y_1, \cdots, y_n ; y_1', \cdots, y_n')dx$$

involving n unknown functions $y_1, \cdots, y_n$ of x and their first partial derivatives $y_1', \cdots, y_n'$, connected by $r < n$ differential equations $f_\rho(x, y_1, \cdots, y_n ; y_1', \cdots, y_n') = 0$ $(\rho = 1, 2, \cdots, r)$.

F. N. COLE,
Secretary.

THE NEW HAVEN COLLOQUIUM.

THE Fifth Colloquium of the AMERICAN MATHEMATICAL SOCIETY was held, at the close of the thirteenth summer meeting, at Yale University, New Haven, Conn., opening on Wednesday morning, September 5, 1906 and extending until noon of the following Saturday. Since the colloquium has become a highly important element in the Society's activities, an outline of its historic development may here be of interest.

The first colloquium organized by the Society was held in connection with the third summer meeting at Buffalo, N. Y., September 2–5, 1896. Two courses of six lectures each were delivered before an audience of thirteen members. Professor Maxime Bôcher considered "Linear differential equations and their applications," and Professor James Pierpont discussed "The Galois theory of equations."*

The second colloquium was held at the close of the fifth summer meeting, at Cambridge, Mass., August 22–27, 1898, before twenty-six persons. Professor W. F. Osgood delivered six lectures on "Selected topics in the general theory of functions," and Professor A. G. Webster six lectures on "The partial differential equations connected with wave propagation."†

The third colloquium, held in connection with the eighth summer meeting, at Ithaca, N. Y., August 21–24, 1901, was attended by twenty-five persons. Two courses of four lectures each were delivered: by Professor Oskar Bolza on "The simplest type of problems in the calculus of variations," and by Professor E. W. Brown on "Modern methods of treating dynamical problems and in particular the problem of three bodies."‡

The fourth colloquium was held immediately after the tenth summer meeting at Boston, Mass., September 2–5, 1903, before an audience of thirty-one members. Three courses of lectures were given: by Professor E. B. Van Vleck, "Selected topics in the theory of divergent series and continued fractions," six lectures; Professor H. S. White, "Linear systems of curves on algebraic surfaces," three lectures; Professor F. S. Woods, "The connectivity of non-euclidean space," three lectures.§

*See the report by Professor T. S. Fiske, BULLETIN, series 2, volume 3, pages 49–59. Professor Bôcher's lectures were published in part in the *Annals of Mathematics*, series 1, volume 12 (1898), pages 45–53. Professor Pierpont's lectures were published in the same journal, series 2, volume 1 (1899), pages 113–143 and volume 2 (1900), pages 22–56.

†See the report by Professor H. S. White, BULLETIN, volume 5, pages 57–58. Professor Osgood's lectures were published in the same volume, pages 59–87. The lectures of Professor Webster are to be published soon.

‡See the report by Professor Edward Kasner, BULLETIN, volume 8, pages 22–25. Professor Bolza's lectures are embodied in his Lectures on the Calculus of Variations, University of Chicago Decennial Publications, No. 14. An abstract of Professor Brown's lectures was published in the BULLETIN, volume 8, pages 103–113.

§See the report by Professor F. N. Cole, BULLETIN, volume 10, pages 119–120, and an abstract of Professor White's lectures immediately following. All the courses have since been published for the Society by the Macmillan Company under the title The Boston Colloquium. This book is reviewed by Professor J. I. Hutchinson in the present number of the BULLETIN.

At the December meeting 1905, the Council appointed a committee consisting of Professors J. Pierpont, P. F. Smith, H. Maschke, H. S. White and F. N. Cole to arrange for a fifth colloquium. A preliminary circular announcing the general features was issued in June. The colloquium opened on Wednesday morning, September 5, 1906, in the lecture rooms of Lampson Hall, Yale University, the following 43 persons being in attendance:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor E. W. Brown, Dr. A. B. Chace, Dr. J. E. Clarke, Dr. A. Cohen, Mr. H. H. Conover, Professor D. R. Curtiss, Mr. H. N. Davis, Professor L. P. Eisenhart, Professor W. B. Fite, Professor A. S. Gale, Professor C. N. Haskins, Dr. L. I. Hewes, Professor Edward Kasner, Professor O. D. Kellogg, Dr. W. R. Longley, Mr. E. B. Lytle, Professor James McMahon, Professor T. E. McKinney, Professor H. P. Manning, Professor Max Mason, Mr. E. A. Miller, Professor E. H. Moore, Professor James Pierpont, Professor H. L. Rietz, Dr. R. G. D. Richardson, Miss S. F. Richardson, Mr. A. R. Schweitzer, Miss I. M. Schottenfels, Professor C. S. Slichter, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor W. E. Story, Professor E. B. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor E. J. Wilczynski, Professor T. W. D. Worthen, Professor J. W. Young.

Three courses of lectures were given, as follows:

Professor E. H. Moore: "On the theory of bilinear functional operations." Five lectures.

Professor E. J. Wilczynski: "Projective differential geometry." Four lectures.

Professor Max Mason: "Selected topics in the theory of boundary value problems of differential equations." Four lectures.

Two lectures were given each morning, two on each of the afternoons of Wednesday and Friday, and one on Wednesday evening, the lecturers alternating. Printed syllabi of all the courses had been issued in advance, which greatly aided intelligent participation. Thursday afternoon was devoted to a trip on Long Island Sound; in the evening the members dined together at Momauguin. The remaining evenings were devoted to social conference at the Graduate Club, whose privileges were extended to members of the Society. The hospitality of the Club,

of Yale University, and particularly of the members of the mathematical department were gratefully acknowledged by a unanimous vote of thanks and appreciation at the closing meeting.

Detailed reports of the courses, prepared by the lecturers, will appear in later numbers of the BULLETIN.

VIRGIL SNYDER.

THEORY AND CONSTRUCTION OF TABLES FOR THE RAPID DETERMINATION OF THE PRIME FACTORS OF A NUMBER.*

BY PROFESSOR ERNEST LEBON.

BY making use of some hitherto unnoticed properties of certain arithmetic progressions, I have succeeded in constructing a table giving very rapidly the solution of the following double problem: *To determine whether a given number is prime or composite, and in the latter case to find its prime factors.* The process which I employ is applicable to large numbers.†

1. Let B be the product $\alpha\beta\cdots\lambda$ of the consecutive prime numbers α, β, $\cdots$, λ, beginning with 2; P the product $(\alpha-1)(\beta-1)\cdots(\lambda-1)$; I any of the P numbers that are relatively prime to B and less than B; K a number successively equal to the positive integers, starting from zero.

We easily see that *the system of P arithmetic progressions whose general term is $BK+I$ contains all the prime numbers except those that occur in B.*

We shall say that B is the base of the system and that I is the index of a term of this system.

Two indices will be said to be complementary when their sum is equal to the base.

2. Let N, D and M be any numbers relatively prime to B. In order to avoid ambiguity, I will write D in the form $BK'+I'$.

It is evident that $N(=BK+I)$ *is or is not divisible by D according as K and M do or do not satisfy the equation*

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* Translated by Professor W. B. FITE.

† Cf. *Comptes rendus*, vol. 151 (1905), p. 78. See also § 10, p. 77.

$$(a) \qquad BK + I = MD,$$

B, I and D being known.

3. Let k and m be the minimum values of K and M satisfying equation (a), and n a number successively equal to the positive integers starting from zero. If necessary for clearness, I use k_I for the numbers K relative to a divisor D.

The equality

$$K = k + nD$$

gives the values of K to which correspond all the numbers N that are divisible by D.

From this equality we get the formula

$$(1) \qquad n = \frac{K - k}{D},$$

where K is the integral quotient obtained by dividing N by B; the remainder in this division is the value of I.

We see that *according as the value of n obtained by applying formula* (1) *is integral or fractional, the number N is, or is not, a multiple of the divisor D.*

Then *the table of numbers k set up for a system of base B enables one to recognize whether N is prime or not by dividing the difference $K - k$ by the prime numbers less than $\sqrt{N}$ and greater than λ; if N is not prime, this procedure gives its prime factors.*

We see that the larger the base B the more rapidly this method gives the result.

Before applying formula (1), it should not be forgotten that if we are considering a number N' we must in order to get N remove from it the factors that are common to it and B.

4. The numbers k I shall call *characteristics.*

5. In order to find methodically and quickly the characteristics k which correspond to the P arithmetic progressions of a system with the base B, we can use the following formula, which is obtained by replacing in equation (a) K and M by k and m, and D by $BK' + I'$:

$$(2) \qquad k = \frac{I'm - I}{B} + K'm.$$

Formula (2) gives the characteristic k when the value of m is such that the binomial $I'm - I$ is divisible by B.

6. The three following theorems, which are easily demonstrated, enable one to make a considerable reduction in the number of operations required for the calculation of the characteristics k:

I. *To the product $I'm$ of the two indices I' and m correspond an index I and a characteristic k; this characteristic is associated with the number $I'm$ by the arithmetic progression of base B and index I given by this product.*

II. *The P arithmetic progressions of a system of base B being arranged in the order of the increasing values of the indices I of their terms, the sum of the two characteristics k and that of the two values of m relative to the same divisor D and to two progressions equidistant from the extremes are equal to $D-1$ and B respectively.*

III. *If the values of I, k_I, I', and m satisfy the equation*

$$Bk_I + I = I'm,$$

and if we consider the equation

$$Bk_{B-I} + (B - I) = (B - I')m,$$

where the two indices $B - I$ and $B - I'$ are complementary to the I and I' respectively of the preceding equality, the unknown characteristic k_{B-I} is given by the formula

$$k_{B-I} = m - 1 - k_I.$$

7. It follows from Theorems II and III that in order to calculate the binomial $I'm - I$, it is sufficient to associate with the first half of the P values of I' the first half of the P values of m, arranged in the order of magnitude.

The remainder obtained by dividing $I'm$ by B is the index I relative to a progression of the system of base B.

When K' is zero, the first term of formula (2) gives, in each of the P progressions of base B, the P characteristics k corresponding to the P values of I.

Inasmuch as the characteristics k corresponding to the index I are the same when D is equal to either I' or m, it follows from Theorem I that it is sufficient to take the products $I'm$

starting from the value of m equal to the value of I'; that is to say, it is sufficient to take the values of $I'm$ starting from I'^2. We know that we apply the first term of formula (2) only to the values of m which are equal to the first $P/2$ indices. Moreover to the products of 1 by the indices correspond characteristics k which are evidently zero.

Consequently, among the P^2 characteristics k relative to the P divisors which equal the indices there are at most $P(P-2)/8$ characteristics whose determination requires a multiplication and a division.

8. As to the P characteristics k relative to a divisor D superior to $B-1$ and with index I', we can deduce them immediately from the P characteristics found for $D = I'$ by making use of the last term of formula (2).

9. In order to apply formula (1), we can make use of a table of characteristics relative to the base B containing at the top of the columns only the first half of the P indices I arranged in order of magnitude, and below each index I_n the complementary index $B - I_n$; then in these columns, in regard to the prime divisors D, the values of k relative to the first half of the P indices I.

Then, having a number N which does not contain any of the prime factors of B, we divide N by B. This gives the quotient K and the remainder I, which I shall call I_n if it belongs to the first half of the P indices I arranged in order of magnitude, and $I_{n'}$ if it belongs to the second half of these indices.

When the remainder is I_n, the index is also I_n and the characteristic k is equal to the value k_n given in the table.

According as D is, or is not, a multiple of the difference $K - k_n$, D is, or is not, a prime divisor of the number whose index is I_n or of the number whose index is $I_{n'}$.

10. The table of characteristics relative to the base 30030, with the prime divisors from 17 to 30029 enables one to solve the problem in question between 1 and 30030^2 or 901800900.

Suppose that the table of characteristics k relative to the base B is formed of columns headed by all the indices I in order of magnitude and of rows headed by the prime divisors D arranged in the order of magnitude. The characteristic k corresponding to a number N of index I_n and to a prime divisor D is found at the intersection of the column I_n and the row D.

Let N be a number of the form $30030K + I$. In making

the trial we will stop at the prime divisor D_n immediately inferior to $\sqrt{N}$.

We consider whether K is equal to one of the characteristics which correspond to the index I; for this it is sufficient to start from the prime divisor immediately superior to K.

When K is equal to one or several of these characteristics, N admits prime divisors which correspond to these characteristics. Then we have immediately the composition of N.

When K is not equal to any of these characteristics, we form the differences $K - k$ for the prime divisors 17, 19, 23, ···. These differences are always less than 30029, because K is here less than B and k is less than D and hence less than B. A difference $K - k$ is, or is not, equal to an index. In the former case, we recognize without calculation whether the difference $K - k$ is divisible by the divisor that corresponds to it. In the latter case we recognize nearly always whether a difference $K - k$ is divisible by the corresponding divisor D without performing the division; then we decompose $K - k$ into factors, one of which is either one of the prime numbers 2, 3, 5, 7, 11 and 13, or a product of some of these, and the other an index. In most cases it is not necessary to perform this decomposition in order to see if a difference is divisible by the prime divisor which corresponds to it.

If there is no difference $K - k$ that is divisible by any of the prime factors from 17 to D_n, N is prime. If we find a difference $K - k$ that is divisible by the prime divisor D less than D_n, N is divisible by D. We divide N by D, the resulting quotient also by D, and so on. Let N_1 be the last quotient thus obtained. We treat N_1 as we have just treated N, beginning with the prime divisor immediately following D, and we find that N_1 is the product of characteristics or is prime.

11. In order to recognize instantly whether a difference $K - k$ is divisible by the corresponding divisor D, it is sufficient to have, in addition to the table of characteristics relative to the base 30030 up to the divisor 30029, a table of remainders obtained by dividing the consecutive integers from 17 to 30029 by the divisors D; in fact, a difference $K - k$ is divisible by the corresponding divisor D, when the values of R and of k which correspond to this division are equal.

PARIS,
May, 1906.

ON A FUNDAMENTAL RELATION IN ABSTRACT GEOMETRY.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, September 3, 1906.)

1. In his Abstrakte Geometrie, page 10, Professor Vahlen seeks to define planar order with the aid of two relations, each of which is between a point and a set of linearly ordered points, such that if bc and b_1c_1 are any two pairs of distinct points of the set, b precedes (follows) c and b_1 precedes (follows) c_1 simultaneously. The linear order referred to is established by means of the undefined relation due to Vailati which may be expressed concretely by "precede." Aside from the fact that Vahlen employs a superfluity of undefined symbols, there is a noticeable lack of precision in his presentation.* Recent investigations of the writer have emphasized the importance, in abstract geometry, of a relation used implicitly by Vahlen, which, however, seems to have escaped him entirely. The object of the present note is briefly to point this out.

2. The relation referred to above is $aR(bc)$, which may be considered an abstract statement for the following: "If a person swims from b to c the point a is at his right." In the relation $aR(bc)$, (bc) is an ordered dyad. That is, if $b \neq c$, then the dyads (bc) and (cb) are distinct, and if the points bc are not the points b_1c_1, then the dyads (bc) and (b_1c_1) are also distinct. Thus if (bc) and (b_1c_1) are identical, ordered dyads, then $b = b_1$ and $c = c_1$. The existence of an ordered dyad, it will be observed, does not properly involve any geometric assumptions, except the existence of points.

3. We can now construct planar axioms and definitions which will clarify the Vahlen text.

1. Axioms.

(1) There is at least one point.

(2) If there exists a point, there exist the points a_0, b_0, c_0 such that $a_0R(b_0c_0)$ or $a_0R(c_0b_0)$.

(3) If $aR(bc)$, then $a \neq b, c$.

*Cf. the review by Professor Veblen, Bulletin, July, 1906, p. 505.

(4) If $aR(bc)$ and $b \neq c$, then $a\bar{R}(cb)$.*
(5) If $aR(bc)$ and $x \neq a$, then $xR(bc)$ or $xR(ca)$ or $xR(ab)$.

2. Definitions.

(1) The point x is *collinear* with (ab) if $a \neq b$, $x\bar{R}(ab)$, and $x\bar{R}(ba)$.

(2) The point x is in the *interior* of (ab) if
there is a point c' such that $c'R(ab)$,
$x\bar{R}(ab)$, $x\bar{R}(ba)$,
for *any* point c such that $cR(ab)$ and $xR(bc)$ or $xR(ca)$ follows $xR(bc)$, $xR(ca)$.

(3) The point x is in the *plane* (abc) if $aR(bc)$ and either $xR(bc)$ or $xR(ca)$ or $xR(ab)$.

(4) The point x is in the *interior* of (abc) if $xR(bc)$, $xR(ca)$, and $xR(ab)$.

It will be noticed that it is not necessary to introduce an additional relation $aL(bc)$ as Vahlen does. This relation is expressed by $aR(cb)$. The definitions (1)–(3) are not given by Vahlen.

4. As a consequence of the preceding axioms we prove the following (Cf. Vahlen, Axiom, § 21, page 10).

Theorem 1. If $cR(ab)$, then $aR(bc)$ and $bR(ca)$.

Since $cR(ab)$, by axiom (3), $c \neq a, b$. Hence by axiom (5), $aR(ab)$ or $aR(bc)$ or $aR(ca)$. Therefore by (3), $aR(bc)$. Similarly, $bR(ca)$.

Theorem 2. If $xR(ab)$ and $xR(bc)$ and $xR(ca)$, then $cR(ab)$.

We may assume $c \neq x$. Then by axiom (5), $cR(ab)$ or $cR(xa)$ or $cR(bx)$. By theorem 1, if $cR(xa)$ or $cR(bx)$, then $xR(ac)$ or $xR(cb)$ respectively. Hence by axioms (3), (4), and the hypothesis, $cR(ab)$ is the only possibility. The theorem therefore follows.

5. It would be easy to construct further axioms so that the preceding definitions are effective. For this purpose it will be most convenient to refer to the writer's planar descriptive system.† There results, then, that by using two undefined symbols, point and the (planar) relation $aR(bc)$, it is possible to construct a set of axioms for the plane without introducing an undefined linear relation as Vahlen has done. However, the relation $aR(bc)$ or, as we will now write it, $aR_2(bc)$ may be considered an extension, dimensionally, of the undefined relation

* The rule over the R is a symbol of negation.
† Cf. Bulletin, June, 1906, p. 438. For n-ad read $(n+1)$-ad.

of Vailati, which we will write aR_1b. More generally, it is possible to construct a set of n-dimensional axioms ($n = 1, 2, 3, \cdots$) by means of the two undefined symbols point and the n-dimensional relation $aR_n(b_1b_2\cdots b_n)$, where the n-ad is ordered.

CHICAGO,
July, 1906.

ON THE ORDERLY LISTING OF SUBSTITUTIONS.

BY PROFESSOR D. N. LEHMER.

(Read before the San Francisco Section of the American Mathematical Society, February 24, 1906.)

1. In a substitution such as $\left(\begin{smallmatrix}12345\\34251\end{smallmatrix}\right)$ we shall speak of 34251 as the *permutation* belonging to the substitution. The following list of the permutations belonging to the substitutions on four letters exhibits what seems to be the most orderly arrangement:

$1 = 1234 = \mathrm{I}$	$13 = 3124 = (132)$
$2 = 1243 = (34)$	$14 = 3142 = (1342)$
$3 = 1324 = (23)$	$15 = 3214 = (13)$
$4 = 1342 = (234)$	$16 = 3241 = (134)$
$5 = 1423 = (243)$	$17 = 3412 = (13)(24)$
$6 = 1432 = (24)$	$18 = 3421 = (1324)$
$7 = 2134 = (12)$	$19 = 4123 = (1432)$
$8 = 2143 = (12)(34)$	$20 = 4132 = (142)$
$9 = 2314 = (123)$	$21 = 4213 = (143)$
$10 = 2341 = (1234)$	$22 = 4231 = (14)$
$11 = 2413 = (1243)$	$23 = 4312 = (1423)$
$12 = 2431 = (124)$	$24 = 4321 = (14)(23)$

2. It is seen that the numbers of four digits expressed by the different permutations are in ascending order of magnitude. This order would be reversed by the interchange of 1 with 4 and 2 with 3.

3. It is easily possible to compute the rank of a given permutation or to write down a permutation of given rank. Thus the rank of the permutation 341562 is 240 plus the rank

of 41562, there being 120 beginning with 1 and 120 beginning with 2. The rank of 41562 is 48 plus the rank of 1562, since there are 24 beginning with 1 and 24 beginning with 2. The rank of 1562 is the same as the rank of 562 or, since our list gives the rank of any permutation on four elements, we can obtain the rank of 1562 from it as 4. The rank of 341562 is therefore 292. The finding of a permutation of given rank is not more difficult.

4. The permutation of rank 292 in seven letters is 1452673 obtained by prefixing unity and increasing each digit by unity. Similarly for eight letters the permutation of rank 292 is 12563784. The cyclic notations for the corresponding substitutions are

For six letters (13)(2456),
For seven letters (24)(3567),
For eight letters (35)(4678).

In accordance with this rule the multiplication table for the symmetric group on n letters will contain in the upper left hand corner the multiplication table for the symmetric group on $n-1$ letters.

5. The product of two substitutions S and S' in the order SS' is obtainable by operating on the permutation of S with the substitution S', as is readily seen by changing the order of the elements in the upper line and accordingly in the lower line of S'. This law enters in a remarkable way in the multiplication table of the symmetric group. One obvious result is seen in the last column under the substitution (14) (23). This, as we noted in § 2, reverses the order of the permutations so that we have in that column the substitutions in descending rank reading downward. In symbols $S_i S_{24} = S_{25-i}$, or since the permutation of rank $n!$ reverses the order of the arrangement we have generally

$$S_i S_{n!} = S_{n!-i+1}.$$

6. Suppose now that $S_i S_j = S_k$. Then

$$S_i S_j S_{n!} = S_k S_{n!},$$

whence by the above formula

$$S_i S_{n!-j+1} = S_{n!-k+1}.$$

From this it follows that two entries in the table which are on the same horizontal line and are equally distant from the ends add up $n! + 1$. Thus $12 \cdot 17 = 21$ and $12 \cdot 8 = 4$ and $21 + 4 = 25 = 4! + 1$.

7. Suppose again that

$$S_{n!}S_j = S_k.$$

Then

$$S_i S_{n!} S_j = S_i S_k,$$

or

$$S_{n!-i+1} S_i = S_j S_k.$$

From this formula it follows that the column in the table with k for its lowest element reads upward in the same way that the column with k for its topmost element reads downward. These two curious laws hold in their entirety for any subgroup of the symmetric group which contains the substitution of rank $n!$ This in particular will be the case with the alternating group when n is of the form $4m$ or $4m + 1$, since in these two cases the substitution $S_{n!}$ is an even one. The metacyclic group of order $p(p-1)$ will also exhibit these laws since it contains the substitution $\left(\begin{smallmatrix} x \\ x^{p-1} \end{smallmatrix}\right)$ which is of rank $n!$

8. The list of the permutations of the substitutions on n letters divides itself naturally into n sets. The first set comprises those beginning with the element 1, the second those beginning with the element 2 and so on. These sets we will call the primary sets. Each of these primary sets is divided into secondary sets $n - 1$ in number, each secondary set beginning with the same two numbers. These secondary sets divide up further into ternary sets and so on. We will examine the arrangement of these sets in the multiplication table of the symmetric group.

9. In the product SS' let S' remain fixed while S runs through the list of $n!$ substitutions in order. This by § 5 amounts to operating on all the permutations in order with the substitution S'. It is clear that none of the sets are separated by this operation, the substitution S' merely interchanging the sets as wholes and permuting the subsets among themselves. Moreover the permutation of the primary sets will be of the same rank as the permutation of S'. Thus in the column headed 10 in the group on four elements it is seen that the primary sets are listed reading down in the order 2341 which is the permutation of rank 10.

10. Consider now any primary set beginning with the element k. The secondary sets in it are clearly permuted among themselves in accordance with the effect of S' on all the elements except k. The permutation of the secondary sets may then be obtained by striking out k from the permutation of S'. Thus in column 10 as before referred to the order of the secondary sets in the first primary set is 341; in the second primary set is 241; in the third primary set is 231, and in the fourth 234. The order for the ternary sets is similarly obtained so that for instance the ternary sets in the first primary set have the orders 41, 31, 34, obtained by striking out 3, 4 and 1 in succession from the order of the secondary set 341.

The multiplication table for the symmetric group on four letters is appended; also the table for the alternating group.

Symmetric Group.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
8	7	14	13	20	19	2	1	16	15	22	21	4	3	10	9	24	23	6	5	12	11	18	17
9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
13	19	7	20	8	14	15	21	1	22	2	16	9	23	3	24	4	10	11	17	5	18	6	12
14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
20	14	19	8	13	7	22	16	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	7	15	1	9	3
23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

Alternating Group.

1	4	5	8	9	12	13	16	17	20	21	24		13	20	8	21	1	16	9	24	4	17	5	12
4	5	1	12	8	9	16	17	13	24	20	21		16	24	12	20	4	17	8	21	5	13	1	9
5	1	4	9	12	8	17	13	16	21	24	20		17	21	9	24	5	13	12	20	1	16	4	8
8	13	20	1	16	21	4	9	24	5	12	17		20	8	13	16	21	1	24	4	9	12	17	5
9	17	21	5	13	24	1	12	20	4	8	16		21	9	17	13	24	5	20	1	12	8	16	4
12	16	24	4	17	20	5	8	21	1	9	13		24	12	16	17	20	4	21	5	8	9	13	1

THE BOSTON COLLOQUIUM.

Lectures on Mathematics. By E. B. VAN VLECK, H. S. WHITE, F. S. WOODS. New York, The Macmillan Company, 1905. xii + 187 pp.

THIS volume contains the lectures delivered at the Boston Colloquium of the AMERICAN MATHEMATICAL SOCIETY, September 2–5, 1903; it is not only a convenient form in which to preserve these valuable lectures, but is a most welcome addition to the meagre catalogue of English works on the higher mathematics. In the brief space of 166 pages is given a very clear and readable summary of three important fields of recent development. The topics treated have the following titles:—

Linear systems of curves on algebraic surfaces, by Professor White, pages 1–30;

Forms of non-euclidean space, by Professor Woods; pages 31–74;

Selected topics in the theory of divergent series and of continued fractions, by Professor Van Vleck, pages 76–166, followed by a bibliography of memoirs relating to algebraic continued fractions, pages 167–187.

The authors have displayed admirable judgment in selecting from the abundance of material at their disposal those theorems and results most suited to give a distinct and vivid picture of the fields with which it is intended to make the reader acquainted. A judicious and carefully considered reserve is exercised in the presentation of details, some of the theorems being demonstrated in full, while for others only an outline is given sufficient to present a clear idea of the argument and the essential nature of the method, and still other results are merely stated when their character and bearing can be made evident without the details of proof. The value and suggestiveness of the lectures are augmented by the various critical remarks scattered throughout the work, which serve not only to elucidate and emphasize the relationships of the various parts, but also to indicate their deeper and less obvious import. Then, too, the pointing out of unsolved problems and incomplete features, whether of detail or theory, will at least be of service in stimulating the constructive imagination of the reader, even

if it does not lead to additional contributions in these particular lines.

The first group of lectures deals with the theory of systems of curves on an algebraic surface, with especial reference to properties that are invariant under birational transformations and the kinds of surfaces that admit given systems. This is followed by an account of Picard's theory of linear exact differentials of the first kind on an algebraic surface, a subject of great interest not only from the geometric point of view but also on account of its connection with the slowly developing theory of functions of two variables.

The second set of lectures begins with a clear and simple account of the Riemann hypotheses relative to a restricted portion of space. By the introduction of additional hypotheses the geometry is extended to an unlimited space. With restriction to space of constant curvature three essentially distinct types of geometry are considered, the spherical and elliptic, the lobachevskian, and the euclidean, — these corresponding to positive, negative, and zero curvature respectively. In the first two cases the non-euclidean spaces investigated by Clifford, Klein, and Killing are considered in detail.

The last group of lectures occupies considerably the greater half of the book and thus is afforded a correspondingly larger scope for breadth and variety of treatment, which has been utilized by the author in a most thorough and scholarly manner. The subject, too, is of peculiar interest, since until recent years the divergent series has been looked upon by mathematicians as a monstrosity to be carefully excluded from the well-regulated society of exact science. As, however, the solutions of many differential equations present themselves in the form of divergent series, this subject has during the past twenty years made good its claims to serious recognition, and, thanks to the labors of Poincaré, Stieltjes, Borel, and others, a substantial foundation has been laid for a theory of divergent series. To determine the conditions under which a divergent series may be manipulated as the analytic representative of an unknown function, to develop the properties of the function, and to formulate methods for deriving a function uniquely from the series constitute the chief aim of the theory and form the subject matter of the lectures. The greater share of discussion is given to the last named object. The attention is here directed towards convergent series of rational functions, and the valuable

instrumentality of algebraic continued fractions leading to a somewhat full and independent discussion of the theory of such fractions.

A few typographical errors have been observed, none of which would be confusing to the reader. The formulas in x, pages 33–34, should, of course, be expressed in terms of z. The word *sixth*, page 29, line 20, is apparently incorrect, as Humbert states explicitly (*Liouville*, 1893, page 436) that the minimum degree of hyperelliptic surfaces is not yet determined, but he believes it to be *eight*, and considers a number of cases of surfaces of that degree (pages 436–449).

The book is unfortunately printed on very thick paper; while not bulky, it could have been made into a more tasty and compact volume of less than half the thickness.

J. I. HUTCHINSON.

CORNELL UNIVERSITY.

CORRECTION.

The following correction should be made in the paper by Mr. Lennes in the October BULLETIN: Page 14, lines 14–16, *for* where M is the difference $\cdots$ of $f(x)$ on ab *read* where M is twice the least upper bound of the absolute value of $f(x)$ on ab.

NOTES.

BY arrangement with The Macmillan Company a limited number of copies of the Chicago Congress Mathematical Papers are offered to members of the AMERICAN MATHEMATICAL SOCIETY at $1.50 per volume, one half the former price. Orders should be sent direct to the Macmillan Company or to the Secretary of the Society, no commission being allowed to agents. Details concerning the book will be found in the advertising pages of the BULLETIN.

THE seventy-sixth annual meeting of the British association for the advancement of science was held at York, England, August 3 to 8. Professor R. LANKESTER was president of the association, and Dr. E. H. GRIFFITHS president of section A,

mathematics and physics. The papers of mathematical interest were: "On the notation and use of vectors," by Professor O. Henrici; "On the irregular motion of the earth's pole," by Mr. E. H. Hills and Professor J. Larmor; "Expansions in products of oscillatory functions," by Professor A. C. Dixon; "On the motion of a particle in a cyclone," by Professor W. H. H. Hudson; "On hypereven numbers," by Lt. Col. A. Cunningham; "On the Lagrange equation," by Professor A. R. Forsyth; "Two new symmetric functions," by Major P. A. MacMahon; "Finite groups," by Mr. H. Hilton; "On multiple series," by Professor T. J. I'A. Bromwich; "Many valued functions of real variables," by Mr. A. R. Richardson; "On methods of computing Bessel functions for large values of the argument," by Professor A. Lodge. The next meeting is to be held at Leicester, that of 1908 at Dublin, and in 1909 the association will for the third time visit Canada, the meeting to be held in Winnipeg.

THE Royal Belgian academy of sciences announces the following prize problems in astronomy and mathematics for the year 1907 (each prize is 800 francs):

(1) To find, in altitude and azimuth, the expressions of the principal terms of the periodic deviations of the vertical, under the hypothesis of the non-coincidence of the centers of gravity of the crust and core of the earth.

(2) Between the elements of two forms of the second order (two non-superposed plane systems, or a plane and a bundle, or two bundles with different vertices) a quadratic correspondence is established (in the sense of Reye, Geometrie der Lage, volume 2, chapter 22). It is desired to study the system of elements obtained by joining or by intersection of pairs of homologous elements in the two forms of the second order.

THE foreign universities below offer courses in mathematics during the year 1906–1907 as follows:

CAMBRIDGE UNIVERSITY. — *Michaelmas term*, 1906. — By Professor A. R. FORSYTH: Abel's theorem and abelian functions, three hours. — By Professor Sir G. H. DARWIN: Theory of potential and attractions, three hours. — By Professor Sir R. S. BALL: Spherical astronomy, three hours. — By Professor J. LARMOR: Electricity and magnetism, three hours. — By Professor B. HOPKINSON: Applied mathematics, two hours. — By

Dr. E. W. HOBSON: Representation of functions by series, three hours. — By Dr. H. F. BAKER: Theory of functions, three hours; By Mr. H. W. RICHMOND: Analytic geometry, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics, three hours. — By Mr. A. BERRY: Elliptic functions, Bessel functions and Fourier series, three hours. — By Mr. A. MUNRO: Hydrodynamics and sound, I, three hours. — By Mr. J. H. GRACE: Invariants and geometric applications, three hours. — By Mr. E. W. BARNES: Taylor's series, three hours.

Lent term, 1907. — By Professor A. R. FORSYTH: Calculus of variations, three hours. — By Professor Sir G. H. DARWIN: Figure of the earth and precession, three hours. — By Professor Sir R. S. BALL: Spherical astronomy, II, three hours. — By Professor J. LARMOR: Electrodynamics, three hours. — By Professor B. HOPKINSON: Applied mathematics, two hours. — By Dr. E. W. HOBSON: Vibrations and sound, three hours. — By Dr. H. F. BAKER: Theory of continuous groups, three hours. — By Mr. H. W. RICHMOND: Analytic geometry, II, three hours. — By Mr. A. N. WHITEhead: Principles of mathematics, II, three hours. — By Mr. B. A. HERMAN: Hydromechanics, I, three hours; Hydrodynamics, three hours. — By Mr. C. T. BENNETT: Line geometry, three hours. — By Mr. E. W. BARNES: Linear differential equations, three hours.

Easter term, 1907. — By Professor J. LARMOR: Theory of gases and thermodynamics, two hours. — By Mr. H. W. RICHMOND: Projective geometry, three hours. — By Mr. A. N. WHITEHEAD: Noneuclidean geometry, three hours. — By Mr. A. YOUNG: Theory of invariants, three hours. — By Mr. G. H. HARDY: Integral functions, three hours. — By Mr. R. R. WEBB: Definite integrals, three hours.

UNIVERSITY OF BERLIN (winter semester.) — By Professor H. A. SCHWARZ: Differential calculus, four hours; synthetic geometry, four hours; Geometric conics, two hours; Seminar, two hours; Colloquium, two hours. — By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours. — By PROFESSOR F. SCHOTTKY: Integral calculus, four hours; General theory of functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Definite integrals, two hours. — By Professor J. KNOBLAUCH: Determinants, four hours; Curved surfaces, four hours; Twisted curves, one hour. — By Pro-

fessor R. LEHMANN-FILHÉS: Analytic geometry, four hours. By Professor E. LANDAU: Theory of numbers, four hours. — By Dr. I. SCHUR: Ordinary differential equations, four hours.

UNIVERSITY OF BONN (winter semester.) — By Professor E. STUDY: Non-euclidean geometry, four hours; Introduction to analytic mechanics, four hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Calculus, II, four hours; Fourier's series, two hours; Geometry of numbers, two hours; Seminar, two hours. — By Professor F. LONDON: Analytic geometry of two and three dimensions, four hours; Descriptive geometry with exercises, three hours; Seminar, two hours. — By Dr. E. SCHMIDT: Introduction to algebra, three hours; Determinants, two hours.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Elements of differential geometry, differential equations and calculus of variations, three hours. — By Professor E. CESÀRO: Mathematical theory of elasticity, three hours. — By Professor L. DONATI: General methods of mathematical physics, Lorentz's theory of electrons, three hours. — By Professor L. PINCHERLE: Elements of algebraic functions and integrals, three hours; Continuous groups of transformations, three hours.

UNIVERSITY OF CATANIA. — By Professor G. LAURICELLA: Functional equations of potential and electrostatics, four and a half hours. — By Professor G. PENNACCHIETTI: Dynamics of solid bodies, II, four and a half hours. — By Professor M. PIERI: Selected chapters of differential geometry, three hours. — By Professor C. SEVERINI: Groups of substitutions and the Galois theory of algebraic equations, three hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Differential and integral equations, developments in series of normal functions, three hours. — By Professor G. LORIA: General theory of algebraic curves and surfaces, three hours. — By Professor O. TEDONE: Vector analysis, kinematics and statics of continua, three hours.

UNIVERSITY OF MESSINA. — By Professor G. BAGNERA: Algebraic functions of two variables, and their integrals, three hours. — By Professor R. MARCOLONGO: Electrodynamics, Lorentz's theory and its applications to optics, three hours. — By Professor V. MARTINETTI: General theory of plane curves, cubic curves, three hours. — By Professor S. ORLANDO: Par-

tial differential equations of mathematical physics, three hours. — By Professor G. VIVANTI: Applications of the theory of contact transformations, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics from 1200 to 1800, four and a half hours. — By Professor A. CAPELLI: Theory of groups and applications to analysis, four and a half hours. — By Professor P. DEL PEZZO: The non-euclidean geometries, four and a half hours. — By Professor A. DEL RE: Algebra of logic, four and a half hours. — By Professor D. MONTESANO: Hypergeometry, four and a half hours; Imaginary elements of geometry, four and a half hours. — By Professor L. PINTO: Electrostatics, three hours. — By Professor F. SIACCI: General theory of dynamical equations, three hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Theory of analytic functions; assemblages and applications, four and a half hours. — By Professor A. FAVARO: History of mathematics in Italy to the sixteenth century, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Electromagnetic fields, dynamics of electrons, four and a half hours. — By Professor A. PADOA: Mathematical logic, one hour. — By Professor G. RICCI: Mathematical physics, elasticity and application to optics, four hours. — By Professor F. SEVERI: Theory of algebraic functions of one and two variables with geometric applications, four hours. — By Professor G. VERONESE: Geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor F. GERBALDI: Theory of functions and applications, three hours. — By Professor G. B. GUCCIA: Theory of plane curves and surfaces, four and a half hours. — By Professor G. TORELLI: Partial differential equations of mathematical physics, four and a half hours. — By Professor A. VENTURI: Theory of the form of the earth, theory of the tides, four and a half hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Mathematical theory of elasticity, three hours. — By Professor F. ASCHIERI: Projective properties of algebraic forms in hyperspace, with applications, three hours. — By Professor L. BERZOLARI: General theory of functions, elliptic functions, three hours. — By Professor E. PASCAL: Theory of transformations of elliptic functions, with application to the quintic

equation; Differential forms of order higher than two, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Foundations of the geometry of hyperspaces, geometry of general algebraic curves, two hours. — By Professor L. BIANCHI: Theory of functions and elliptic functions, four and a half hours. — By Professor U. DINI: Linear differential equations, four and a half hours. — By Professor A. MAGGI: Principles of electricity, four and a half hours. — By Professor P. PIZZETTI: General notions of spherical astronomy and planetary perturbations, three hours.

UNIVERSITY OF ROME. — By Professor G. BISCONCINI: Mathematical theory of elasticity, with applications, three hours. — By Professor G. CASTELNUOVO: Cremona transformations and linear systems of curves, three hours. — By Professor V. CERRUTI: Calculus of variations, three hours.

UNIVERSITY OF TURIN. — By Professor G. MORERA: Hydrodynamics, three hours. — By Professor E. D'OVIDIO: Theory of functions, Jacobi's and Weierstrass's theories of elliptic functions, three hours. — By Professor G. PEANO: Mathematical logic, two hours. — By Professor C. SEGRE: Groups in geometry, three hours. — By Professor C. SOMIGLIANA: Mathematical theory of elasticity and its applications, three hours.

PROFESSOR E. CESÀRO, of the University of Naples, has been appointed professor of analytic mechanics at the University of Bologna.

At the University of Naples, Professor P. DEL PEZZO (higher and differential geometry) and Professor D. MONTESANO (projective and synthetic geometry) have exchanged chairs.

PROFESSORS L. BIANCHI and P. PIZZETTI, of the University of Pisa, and Professor C. SEGRE, of the University of Turin, have been elected members of the Royal institute of Venice.

PROFESSOR F. ENRIQUES, of the University of Bologna, has been elected corresponding member of the Royal academy of the Lincei. Professor P. PAINLEVÉ, of the University of Paris, has been elected foreign member.

THE University of Aberdeen has conferred the honorary degree of doctor of science on Professor G. VERONESE, of the University of Padua.

DR. M. ADLER has been appointed docent in descriptive geometry at the technical school of Vienna, and Dr. K. BOPP to a similar position at the University of Heidelberg.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, has been elected corresponding member of the Royal society of Göttingen.

ON account of his duties connected with the reorganization of the technical school at Turin, Professor V. VOLTERRA, of the University of Rome, has been relieved from lecturing during the present year.

PROFESSOR A. SOMMERFELD, of the technical school at Aachen, has been appointed professor of theoretical physics at the University of Munich.

PROFESSOR G. B. HALSTED, recently of Kenyon College, Ohio, has been appointed professor of mathematics at the Colorado State normal school, Greeley, Colorado.

AT the University of Missouri, Dr. L. D. AMES has been promoted to an assistant professorship of mathematics.

DR. H. A. CONVERSE, recently of the Baltimore Polytechnic Institute, has been appointed professor of mathematics in Davis and Elkins College, Elkins, W. Va.

MR. C. G. SIMPSON has been appointed instructor in mathematics in the State University of Iowa.

DR. C. H. SISAM, of the U. S. Naval Academy, has been appointed instructor in mathematics at the University of Illinois.

MISS M. H. WALBRIDGE has been appointed professor of physics and mathematics at Wells College, Aurora, New York.

DR. A. L. P. WERNICKE, formerly professor of modern languages in the State College of Kentucky, has been appointed instructor in mathematics in Washington University, St. Louis.

PROFESSOR LUDWIG BOLTZMANN, professor of theoretical physics at the University of Vienna, died by his own hand, September 5, at the age of 62 years.

Catalogues of second-hand mathematical works: Alfred Lorenz, Leipzig, Kurprinzstrasse 10, Katalog 165, logic (logic of mathematics; arithmetic, geometric and infinitesimal methods etc.). — Franz Pietzker, Tübingen, Verzeichnis No. 366, mathematics, physics and astronomy. — Galloway and Porter, Cam-

bridge, England, catalogue No. 30, containing 466 titles in mathematics. — Max Weg, Leipzig, Leplaystrasse 1, catalogue No. 103, geodesy, astronomy, mathematics and physics, 2682 titles. — Gustav Fock, Leipzig, Schlossgasse 7–9, Antiquariats-Verzeichnis, No. 288, 4400 titles in mathematics and physics, including the library of the late Professor O. Stolz.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDRADE (J.). Les fonctions angulaires dans la géométrie de l'ajustage. 8vo. (*L'enseignement mathématique* 8, pp. 257–281.)

BARTOLI (E.). See FREYCINET (C. DE).

BERZOLARI (L.). See ENCYKLOPÄDIE.

BRIOSCHI (F.). Opere matematiche, pubblicate per cura del comitato per le onoranze a Francesco Brioschi. Vol. IV. Milano, Hoepli, 1906. 4to. 8 + 418 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu Göttingen, Leipzig, München und Wien sowie unter Mitwirkung zahlreicher Fachgenossen. (In 7 Bänden.) Vol. III: Geometrie, redigiert von W. F. Meyer. Teil 2, Heft 3: H. G. Zeuthen, Abzählende Methoden; L. Berzolari, Allgemeine Theorie der höheren ebenen algebraischen Kurven. Leipzig, Teubner, 1906. 8vo. Pp. 257–455.

FREYCINET (C. DE). Saggio sulla filosofia delle scienze. Analisi; meccanica. Traduzione da E. Bartoli. Bari, 1906. 12mo. 330 pp. M. 3.00

GRÜNFELD (E.). Zur Theorie der linearen Differentialgleichungen. (Progr.) Wien, 1906. 8vo. 20 pp.

GRUNER (P.). Tabellen für die Exponentialfunktion mit negativen Exponenten. Leipzig, Hirzel, 1906. 8vo. 15 pp. M. 1.00

HIESSL (M.). Einige Elemente der vierdimensionalen Geometrie. (Progr.) Stockerau, 1906. 8vo. 22 pp.

HOLTAPPEL (W. T.). See SCHOUTEN (G.).

MEYER (W. F.). See ENCYKLOPÄDIE.

MIRINNY (L.). Isonolyse (résolution générale des équations). Méthode primordiale. Pentisonolyse (résolution générale de l'équation du cinquième degré). (I. Algébrique proprement dite ou par radicaux; II. Exponentielle; III. Trigonométrique; IV. Géométrique; V. Troponomique.) Aperçu sommaire initial. Edition de 1896 élucidée et abrégée. Paris, Marquet, 1906. 16mo. 4 pp., 3 plates.

MOSNAT (E.). Problèmes de géométrie analytique. Vol. I: A l'usage des candidats aux écoles (centrale, ponts et chaussées, mines, polytechnique, normale) et des aspirantes à l'agrégation des jeunes filles. 3e édition. Paris, Vuibert et Nony [1906]. 8vo. 528 pp.

PICARD (E.) et SIMART (G.). Théorie des fonctions algébriques de deux variables indépendantes. Vol. II. Paris, Gauthier-Villars, 1906. 8vo. 6 + 528 pp.

SCHOUTEN (G.). Differentiaalrekening. Bewerkt naar het college van Schouten door W. J. Holtappel. 1. studiejaar. Delft, 1906. 8vo. M. 1.50

SIMART (G.). See PICARD (E.).

VIVANTI (G.). Elementi della teoria delle funzioni poliedriche e modulari. Milano, 1906. 12mo. 8 + 438 pp. M. 2.50

WIRTH, (J.). Ueber die Elementarteiler einer homogenen Substitution. (Diss.) Freiburg, 1906. 8vo. 31 pp.

WIRTINGER (W.). Ueber die Entwicklung einiger mathematischer Begriffe in neuerer Zeit. Wien, Hölder, 1906. 8vo. 20 pp. M. 0.60

YOUNG (W. H.) and YOUNG (G. C.). Theory of sets of points. Cambridge, University Press, 1906. 8vo. 328 pp. Cloth. 12 s.

ZEUTHEN (H. G.). See ENCYKLOPÄDIE.

II. ELEMENTARY MATHEMATICS.

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